

TECHNICAL NOTE

INCORPORATION OF INTERNAL SURFACE RADIANT EXCHANGE IN THE FINITE-VOLUME METHOD

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INTRODUCTION

Patankar [1] gives a background of the finite-volume method (FVM) for solving fluid flow and heat transfer problems. The situation where radiant exchange exists between surfaces internal to the computational domain, however, was not considered by Patankar. The purpose of this analysis is to derive a discretization procedure for the energy equation, for applications involving surface radiant exchange. The new formulation is compared to a more simple, but inaccurate, method of incorporating surface radiation, and its applicability to a large variety of conduction/convection heat transfer problems is discussed.

ANALYSIS

Consider Fig. 1, where control volume P is opaque and control volume E is radiatively transparent. Control volume P has a net radiant heat flux q_r leaving interface e . Because of the radiant heat flux, alternative derivations for the discretization equations for nodes P and E must be made. In this derivation, the procedures and notation of Chapter 4.2 in Patankar [1] are followed and the simple case of steady, one-dimensional conduction and radiation (no heat generation) is considered. The distances between nodal points E and P and W and P are denoted, respectively, by $(\delta x)_e$ and $(\delta x)_w$. Let e and w denote the interfaces between control volumes that are adjacent to P .

An energy balance on interface e states that

$$\frac{k_P}{(\delta x)_{e-}} (T_P - T_e) + \frac{k_E}{(\delta x)_{e+}} (T_E - T_e) - q_r = 0 \quad (1)$$

where k_P and k_E are the thermal conductivities of control volumes P and E , respectively,

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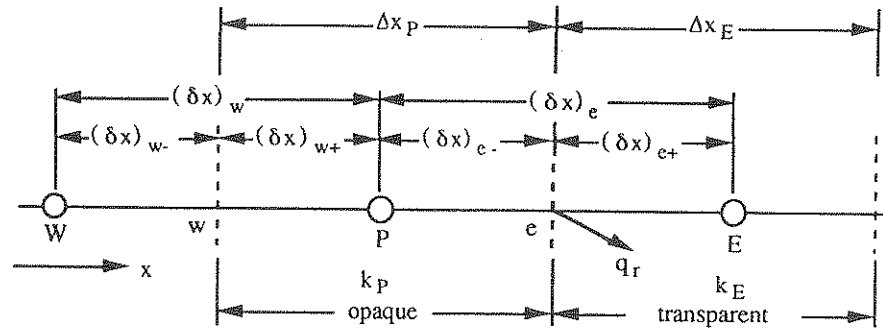


Fig. 1 Control volumes and interfaces.

respectively. T denotes temperature. Note that $(\delta x)_e = (\delta x)_{e-} + (\delta x)_{e+}$. Solving for T_e gives

$$T_e = \frac{[k_P/(\delta x)_{e-}] T_P + [k_E/(\delta x)_{e+}] T_E - q_r}{[k_P/(\delta x)_{e-}] + [k_E/(\delta x)_{e+}]} \quad (2)$$

The energy balance for control volume P states that

$$\frac{k_P}{(\delta x)_{e-}} (T_e - T_P) + \frac{k_w}{(\delta x)_w} (T_w - T_P) = 0 \quad (3)$$

Note that the expression for the heat flux on the east side is based on T_e rather than on T_E . Therefore, the thermal conductivity in this term should be k_P . The interfacial thermal conductivity k_w is calculated from the harmonic mean formula [1]

$$k_w = \left(\frac{f_w}{k_w} + \frac{1 - f_w}{k_P} \right)^{-1} \quad (4)$$

where $f_w = (\delta x)_{w-}/(\delta x)_w$. The expression for T_e from Eq. (2) is inserted into Eq. (3) and Eq. (4) is employed to yield

$$a_P T_P = a_E T_E + a_W T_W + S_{r,P} \Delta x_P \quad (5)$$

The temperature coefficients are given by

$$a_E = \frac{k_E}{(\delta x)_e} \quad a_W = \frac{k_w}{(\delta x)_w} \quad a_P = a_E + a_W \quad (6)$$

and the source term due to radiation is

$$S_{r,P} = -\frac{q_r}{\Delta x_P} \frac{1}{[(\delta x)_{e-}/k_P] \{ [k_P/(\delta x)_{e-}] + [k_E/(\delta x)_{e+}] \}} \quad (7)$$

As a result of the above substitution of the interfacial energy balance, Eq. (2), into Eq. (3), the interfacial thermal conductivity, k_e , in Eq. (6) is given by a harmonic mean formula analogous to Eq. (4). In other words, the inclusion of surface radiant exchange, according to the present analysis, makes it mandatory to utilize the harmonic mean formulation for the interfacial thermal conductivity. A similar derivation for control volume E yields the following radiation source term:

$$S_{r,E} = -\frac{q_r}{\Delta x_E} \frac{1}{[(\delta x)_{e+}/k_E] \{ [k_P/(\delta x)_{e-}] + [k_E/(\delta x)_{e+}] \}} \quad (8)$$

The analysis reveals that the temperature coefficients in the discretization equations do not contain the radiation term and are computed as derived by Patankar [1]. This property of the present formulation should prove to be useful in implementing surface radiant exchange in existing finite-volume computer codes. Radiation appears, however, as source terms in both discretization equations for the nodes adjacent to the radiatively participating interface, as stated by Eqs. (7) and (8). If $k_P = k_E$ and $(\delta x)_{e-} = (\delta x)_{e+}$, the source terms are equal to $-q_r/2 \Delta x$. These terms arise because of the influence of radiation on T_e . Because radiation changes T_e , the conduction fluxes on both sides of interface e are modified. The use of the harmonic mean formulation for the interface conductivities is reflected directly in the radiation source terms and yields realistic results for $k_P \neq k_E$. For example, if $k_E \rightarrow 0$ (i.e., the transparent medium is an insulator), the source terms reduce to $S_{r,P} = -q_r/\Delta X_P$ and $S_{r,E} = 0$, implying that radiation acts as a source for the opaque material only, as it should. Equation (2) allows for the calculation of T_e , which is typically needed in computing the radiative flux q_r on interface e . This needs to be done iteratively because of the nonlinear dependence of Q_r on T_e ; linearization procedures could be worked out, but distract from the general nature of the present formulation.

COMPARISONS

The present formulation for the incorporation of surface radiant exchange is examined using a simple one-dimensional test problem and is compared to (1) the exact solution and (2) a more simple method for the inclusion of radiation. The test problem is illustrated in Fig. 2 and consists of an infinite slab of an opaque medium of thickness ΔX_P and thermal conductivity k_P adjacent to an infinite slab of a radiatively transparent medium of thickness ΔX_E and thermal conductivity k_E . The left and right sides of the domain are isothermal at temperatures T_1 and T_2 , respectively. Heat conduction is one-dimensional and steady. Radiant exchange exists between the interface that separates the transparent and opaque slabs (denoted by e) and the right side of the domain (at T_2). For simplicity, the surfaces are taken as black, so that the net radiative flux leaving surface e , q_r , is given by

$$q_r = \sigma(T_e^4 - T_2^4) \quad (9)$$

where σ is the Stefan-Boltzmann constant. The exact solution for the interface temperature, T_e , can be calculated from

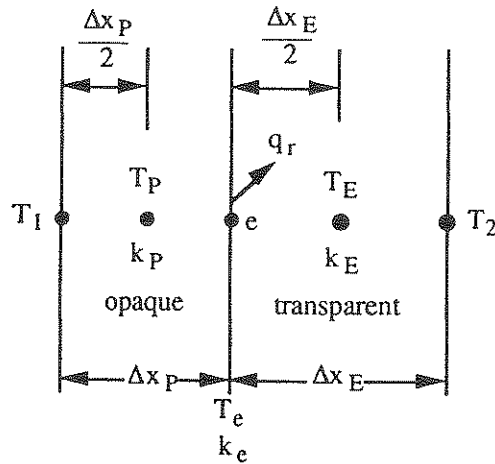


Fig. 2 Test problem.

$$\frac{k_p}{\Delta X_p} (T_1 - T_e) = \frac{k_E}{\Delta X_E} (T_e - T_2) + \sigma(T_e^4 - T_2^4) \quad (10)$$

In the numerical solution of the test problem, a single control volume is assigned to each slab, with the nodal points P and E in the geometric center. A close examination of the present formulation as well as our calculations reveal that the numerical solution always coincides with the exact solution, which is expected.

In order to establish the utility of the present formulation in more detail, it is compared to a more simple method of incorporating surface radiant exchange. This more simple method (1) does not rely on the calculation of the interface temperature T_e [through Eq. (2)], but utilizes the nodal temperature in the opaque slab nearest to the interface (i.e., T_P) in the calculation of the radiative heat flux q_r , and (2) allocates the entire radiative heat flux to the opaque slab. This procedure may be justified by the fact that the radiation is emitted/absorbed by the opaque slab. In the present notation, the more simple method can be expressed as

$$\left. \begin{aligned} T_e &= T_P \text{ in Eq. (9)} \\ S_{r,P} &= -\frac{q_r}{\Delta X_P} \\ S_{r,E} &= 0 \end{aligned} \right\} \quad (11)$$

As before, the harmonic mean formula is utilized for the thermal conductivity at interface e . Note that, in the absence of radiation, the more simple method does reduce to the exact solution.

Comparisons are made for $T_1 = 400$ K, $T_2 = 300$ K, and variable slab thicknesses and thermal conductivities. Results are presented in terms of the relative error in T_P and T_E calculated using the more simple method (recall that the present formulation presented in the previous section coincides with the exact solution for the test problem). Figure 3 shows the error as a function of the slab thicknesses, with $\Delta X_P =$

$\Delta X_E = \Delta X$ and $k_p = k_E = 1.0$ W/m K. As expected, the error in both T_p and T_E increases with increasing slab thicknesses. However, the error in T_E is about one order of magnitude less than the error in T_p , and levels off for $\Delta X > 0.1$ m. Obviously, the magnitude of the error would be different for other choices of T_1 and T_2 , but the trend would be the same.

Figures 4a and 4b show the error in T_p and T_E , respectively, as a function of k_p and k_E with $\Delta X_p = \Delta X_E = \Delta X = 0.01$ m (a typical grid size). In Fig. 4a, the transparent medium is an insulator (e.g., air) with $k_E = 0.01$ W/m K. It can be seen that the error in both T_p and T_E becomes relatively small for a thermal conductivity of the opaque slab, k_p , above 1.0 W/m K, because T_p approaches T_e for a high k_p . On the other hand, for a smaller k_p the error in T_p increases rapidly, indicating that the more simple method of incorporating surface radiant exchange should not be used if the opaque material is an insulator. The error in T_E stays relatively low. This is further underscored in Fig. 4b, where the thermal conductivity of the opaque material is kept at a low value ($k_p = 0.01$ W/m K). For all thermal conductivities of the transparent material k_E , the error in T_p is very large, whereas the error in T_E is relatively low.

In summary, the above calculations clearly indicate the danger of basing the radiative heat flux on the opaque material only, as is done in the more simple method. Unless a very fine grid is utilized in the opaque material in the neighborhood of the radiatively participating interface, a large error may result in the temperatures of the opaque material. On the other hand, an excessively fine grid does not appear to be necessary on the transparent side of the interface. The new formulation presented in this article, however, does not suffer from these shortcomings and coincides with the exact solution for any grid size and thermal conductivities.

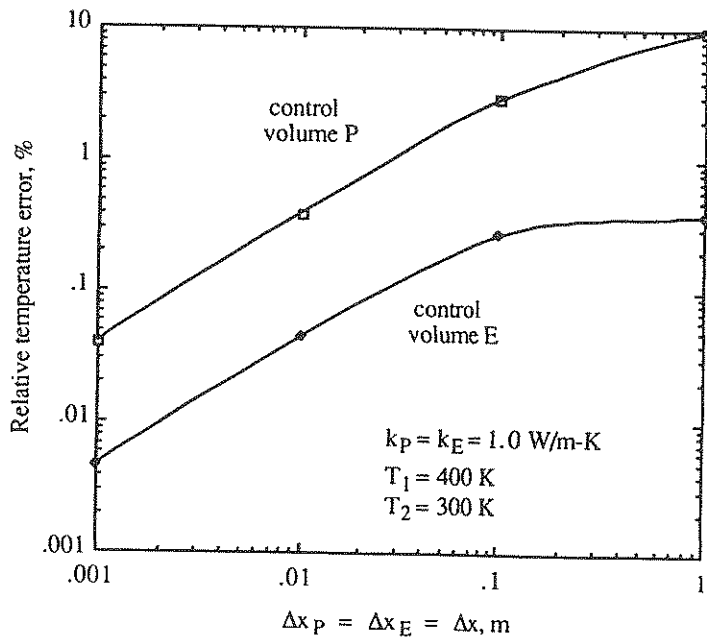


Fig. 3 Effect of grid spacing.

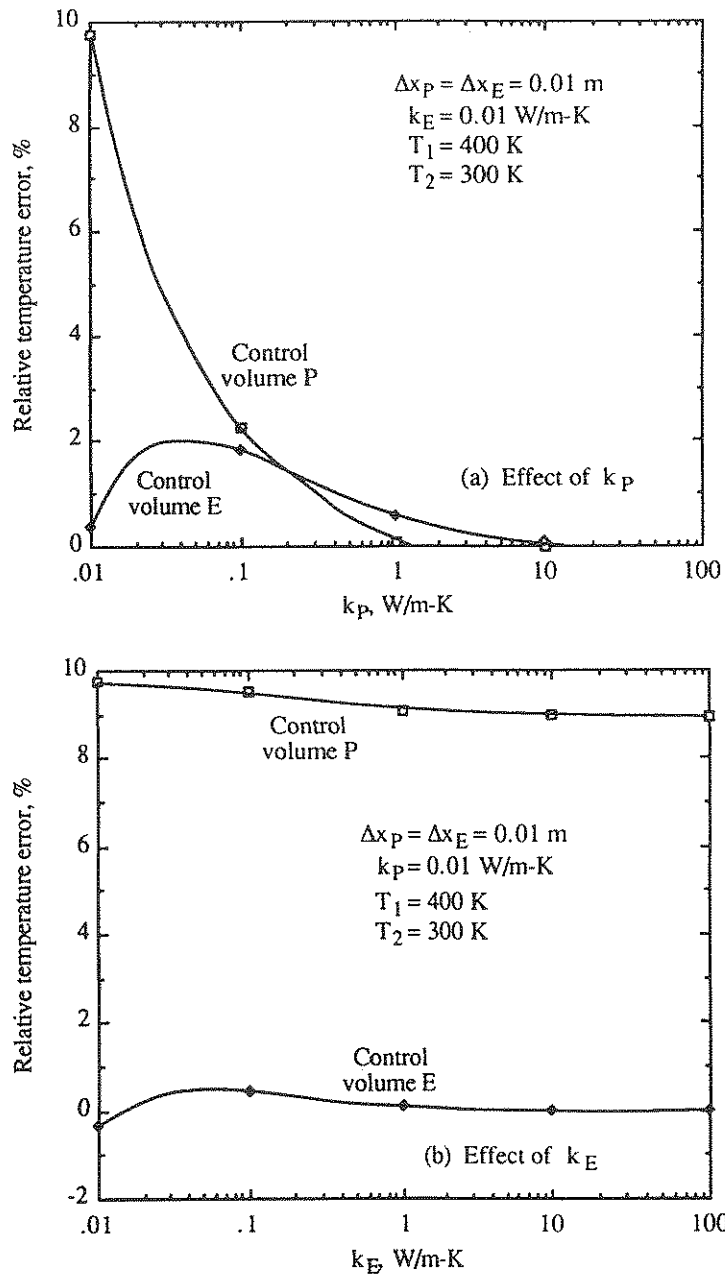


Fig. 4 Effect of thermal conductivity on temperature.

EXTENSIONS

The present formulation for incorporating surface radiant exchange into the finite-volume method was derived for a steady, no-source, one-dimensional situation. Extension of the method to transient, and multidimensional heat conduction with heat generation is straightforward. It can be expected that the inclusion of radiation would not cause additional inaccuracies beyond those associated with the finite-volume method itself. The present formulation is equally valid for conjugate heat transfer problems involving radiant exchange at a solid-fluid interface internal to the computational domain. This is because the mass flux (or velocity) across such an interface vanishes and the temperature profiles between the nodes adjacent to the interface and the interface itself become, in the finite-volume method as described by Patankar [1], linear. In other words, it would be inconsistent to calculate the interface temperature from an expression other than Eq. (2). Application of the present formulation to an electronic cooling problem involving heat generation, conduction, convection, and surface radiant exchange inside a relatively complicated two-dimensional geometry has been demonstrated by Smith et al. [2].

CONCLUSIONS

The analysis of Patankar [1] has been extended to include an internal surface radiant exchange formulation in the numerical solution of conduction/convection heat transfer problems. The radiation terms are accounted for in the source terms, while the temperature coefficients remain unchanged. Consistent and accurate results have been obtained with the present formulation.

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