PERGAMON

International Journal of Heat and Mass Transfer 41 (1998) 4025–4029

International Journal of
HEAT and MASS TRANSFER

Technical Note

Integral solutions of diffusion-controlled dendrite tip growth

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Received 24 November 1997; in final form 3 March 1998

Nomenclature

C specific heat
$E_1(x)$ exponential integral function defined in eqn (2)
$g(x)$ function as defined in eqn (26)
k thermal conductivity
L latent heat
Pe Peclet number
R local radius of revolution of a paraboloid
r radius coordinate
St Stefan number
T_∞ ambient temperature
T_m melting temperature
u_x, u_y velocity components
V tip velocity
x, y coordinates.

Greek symbols
α thermal diffusivity
δ_T thermal boundary thickness
ρ dendrite tip radius
θ dimensionless temperature
φ angle as defined in Fig. 1.

1. Introduction

Dendrites are one of the most important growth forms in solidification, and widely exist in nature and in engineering applications [1]. The dendrite tip can be satisfactorily modeled as a paraboloid of revolution, as originally proposed by Papapetrou in 1935 (see [1]). The first analytical solution for dendritic growth was obtained by Ivantsov [2], by solving the thermal diffusion problem for an isothermal paraboloid growing steadily in a uniformly supercooled melt. The resulting 'Ivantsov function', $Iv(Pe)$, relates the dimensionless supercooling, St, to the tip growth Peclet number, Pe (see below for a definition of the dimensionless parameters) as

$$St = Iv(Pe) = Pe\exp(Pe)E_1(Pe) \tag{1}$$

where $E_1(Pe)$ is the exponential integral,

$$E_1(Pe) = \int_{Pe}^{\infty} \frac{\exp(-x)}{x}\,dx \tag{2}$$

The Ivantsov solution finds extensive applications in crystal growth and other solidification processes. Horvey and Cahn [3] later generalized Ivantsov's analysis to paraboloidal dendrites of arbitrary ellipticity by introducing an eccentricity parameter. The exact solution in eqn (1) can be expressed as an infinite series [4]. For convenience in numerical calculations, several polynomial curve fits have been proposed to approximate the Ivantsov function [4].

The objective of this Technical Note is to develop an alternative approach for solving the classical dendrite tip growth problem. The present approach is based on the approximate integral method. The integral method has been successfully applied in the past to solve boundary-layer flow and transient heat transfer problems, the Stefan problem [5], and other problems that admit no exact solution or where exact solutions are too cumbersome to use in engineering applications [5–7]. In the integral method, the conservation principle embodied in the partial differential equation is not honored locally, but only globally. In the following, it is demonstrated that the integral method can provide very accurate approximations to the Ivantsov solution if appropriate temperature profiles in the thermal boundary layer around the dendrite tip are chosen. First, the integral form of the energy equation is derived. Then, integral solutions for three assumed temperature profiles are presented and compared with the exact solution. Finally, several conclusions are reached based on the present analysis.

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PII: S0017-9310(98)00103-3

2. Analysis and results

Figure 1 depicts the dendrite tip geometry and the coordinate system used in the present analysis. We adopt the same assumptions as in the original work of Ivantsov [2], including purely diffusive transport of heat. The dendrite tip is modeled as a paraboloid of revolution with a tip radius ρ, and assumed to be isothermal at the melting temperature, T_m. The melt far from the dendrite tip is supercooled at a temperature T_∞. The dimensionless supercooling is defined as, $St = (T_m - T_\infty)/(L/C)$, where L and C are the latent heat and specific heat of the liquid, respectively. The tip grows steadily at a velocity V along the negative z-axis, as shown in Fig. 1. The tip growth Peclet number is defined as, $Pe = V\rho/(2\alpha)$, where α is the thermal diffusivity of the liquid. A thermal boundary layer develops around the growing dendrite tip. The boundary layer thickness is denoted by δ_T. The origin of the x–y coordinate system is fixed to the moving tip (see Fig. 1), and x and y are the coordinates tangential and normal, respectively, to the solid/liquid interface. The energy conservation equation for the liquid in this moving coordinate system can be expressed as follows:

$$u_x \frac{\partial \theta}{\partial x} + u_y \frac{\partial \theta}{\partial y} = \frac{\alpha}{r} \frac{\partial}{\partial y}\left(r \frac{\partial \theta}{\partial y} \right). \tag{3}$$

It is subjected to the boundary conditions,

$$\theta = 1 \quad \text{at } y = 0 \tag{4a}$$

$$\theta = 0 \quad \text{at } y \geqslant \delta_T. \tag{4b}$$

In eqn (3), the temperature T is nondimensionalized as

$$\theta = \frac{T - T_\infty}{T_m - T_\infty} \tag{5}$$

and $r = R + y\cos\varphi$, where R is the local radius of revolution and φ is the angle between the x-axis and the z-axis (see Fig. 1). The angle is dependent on x, but independent of y, according to

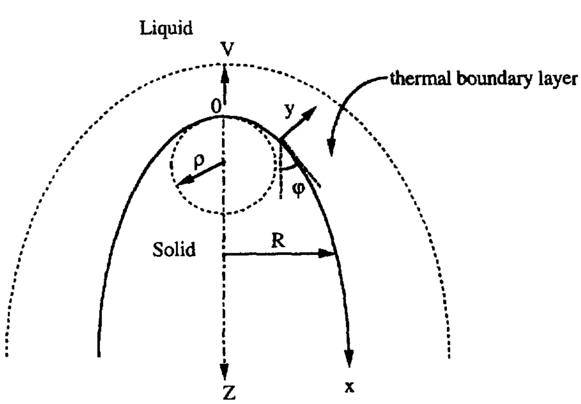

Fig. 1. Schematic of the dendrite tip geometry and thermal boundary layer.

$$\varphi = \tan^{-1}\left(\frac{dR}{dz}\right) = \tan^{-1}\left(\frac{\rho}{R}\right). \tag{6}$$

For diffusion-controlled dendrite tip growth, the velocity components u_x and u_y are given by

$$u_x = V\cos\varphi = \frac{R/\rho}{\sqrt{1+R^2/\rho^2}} V \tag{7a}$$

$$u_y = -V\sin\varphi = -\frac{1}{\sqrt{1+R^2/\rho^2}} V. \tag{7b}$$

It is obvious that thermal diffusion along the x-direction is neglected in eqn (3), as is commonly done in boundary layer analyses [7]. Integrating eqn (3) from $y = 0$ to $y = \delta_T$, we obtain the following integral form of the energy equation,

$$\frac{1}{R}\frac{d}{dx}\left[\int_0^{\delta_T} u_x \theta (R + y\cos\varphi)\, dy \right] = -\alpha \left.\frac{\partial \theta}{\partial y}\right|_{y=0} + u_y|_{y=0}. \tag{8}$$

It is instructive to point out that even if we adopt the concept of a boundary layer, the boundary layer thickness does not have to be thin compared to the local radius of revolution R; in fact, R goes to zero at the dendrite tip. The physical meaning of the various terms in eqn (8) is that the right-hand side represents the sum of thermal diffusion and advective transport from the base of the integral volume, and the left-hand side represents the advective energy transport out of the volume in the x-direction. Since we are only concerned with the dendrite tip where $R \to 0$, and $\varphi \to \pi/2$, the following limits are useful in the analysis,

$$\lim_{R \to 0} \frac{\cos\varphi}{R} = \frac{1}{\rho} \tag{9a}$$

$$\lim_{R \to 0} \frac{dR}{dx} = 1 \tag{9b}$$

$$\lim_{R \to 0} \frac{d\cos\varphi}{dx} = \frac{1}{\rho}. \tag{9c}$$

In order to solve eqn (8) for the boundary layer thickness δ_T, an analytical expression for the temperature profile in the thermal boundary layer needs to be postulated. We present results for three different trial functions for the temperature profile in the present note. Generally, the selection of the trial function requires physical insight and careful consideration of the problem nature. The three trial functions are logarithmic, reciprocal and exponential. They all satisfy the boundary conditions imposed by eqns (4a) and (4b) except for the exponential function, which approaches T_∞ only at infinity. The three trial functions will result in different temperature gradients, hence heat transfer rates at the dendrite tip. As is typical for integral analyses, it is not possible to judge a priori which trial function will lead to the most accurate heat transfer rates and best agreement with the exact Ivantsov solution.

2.1. Logarithmic function

The logarithmic temperature profile is given by

$$\theta = 1 - \frac{\ln(1+y/\rho)}{\ln(1+\delta_T/\rho)}. \quad (10)$$

Substituting eqns (7a), (7b), and (10) into eqn (8), and integrating the left-hand side, we obtain

$$\frac{1}{R}\frac{d}{dx}\left[R\rho V\cos\varphi\left(\frac{\delta_T/\rho}{\ln(1+\delta_T/\rho)}-1\right) + \frac{\rho^2 V\cos^2\varphi}{2}\left(1+\frac{\delta_T/\rho(\delta_T/\rho-2)}{2\ln(1+\delta_T/\rho)}\right)\right] = \frac{\alpha}{\rho}\frac{1}{\ln(1+\delta_T/\rho)} - V\sin\varphi. \quad (11)$$

Due to symmetry, the first derivative of δ_T with respect to x is zero at the tip, i.e.

$$\frac{d\delta_T}{dx} = 0. \quad (12)$$

Then, taking the limit of eqn (11) at the dendrite tip, $R \to 0$, we obtain the expression for the boundary layer thickness δ_T at $x = 0$

$$\frac{\delta_T}{\rho} = \sqrt{1+\frac{1}{Pe}}-1. \quad (13)$$

The Stefan condition at the solid–liquid interface can be written as

$$L(\vec{V}\cdot\hat{n}) = -k(\nabla T\cdot\hat{n}) \quad (14)$$

where k is the thermal conductivity of the liquid. Using eqn (13) to evaluate the heat flux at the interface, eqn (14) results in a relationship between the tip growth Peclet number, Pe, and the tip supercooling, St, as

$$St = Pe\ln\left(1+\frac{1}{Pe}\right). \quad (15)$$

Equation (15) is a good approximation to the exact Ivantsov solution for the full range of supercoolings, $10^{-4} \leqslant St \leqslant 1$, as shown in Fig. 2.

2.2. Reciprocal function

The reciprocal temperature profile can be written as

$$\theta = \frac{1-y/\delta_T}{1+y/\rho}. \quad (16)$$

Substituting eqn (16) into eqn (8) and carrying out the integration, the following expression for the boundary layer thickness at the dendrite tip, δ_T, can be obtained,

$$\frac{\delta_T}{\rho} = \frac{1}{2Pe}. \quad (17)$$

Hence, the approximate solution for this trial function is

$$St = \frac{2Pe}{2Pe+1} \quad (18)$$

or, inversely,

$$Pe = 0.5\left(\frac{St}{1-St}\right) \quad (19)$$

which is exactly the so-called second approximation, I_2, to the Ivantsov function (see [1]). Although eqn (18) is in relatively poor agreement with the exact solution for $Pe < 0.1$, as shown in Fig. 2, it is useful for curve-fitting the inverse Ivantsov function, as was done in Ref. [8]. This can be accomplished by slightly changing the coefficient and the exponential on the parentheses in eqn (19).

2.3. Exponential function

The exponential temperature profile is given by

$$\theta = \exp\left(-\frac{y}{\delta_T}\right). \quad (20)$$

This trial function is based on the one-dimensional steady-state temperature profile ahead of a moving planar front. Substitution into the Stefan condition, eqn (14), yields

$$\delta_T = St\frac{\alpha}{V_n} \quad (21)$$

where V_n is the normal interface velocity given by $V_n = V\sin\varphi$. Upon substitution of eqns (20) and (21) into eqn (8), the integration is carried out to infinity because θ does not approach zero at $y = \delta_T$ (i.e. the boundary layer thickness given by eqn (21) is only a characteristic value at which θ has decreased to e^{-1}). The result is:

$$\frac{1}{R}\frac{d}{dx}\left[R\alpha St\frac{\cos\varphi}{\sin\varphi} + \frac{\alpha^2 St^2}{V}\frac{\cos^2\varphi}{\sin^2\varphi}\right] = \frac{V\sin\varphi}{St} - V\sin\varphi. \quad (22)$$

At the dendrite tip, $R \to 0$, the above equation becomes

$$\left(\frac{1}{St}-1\right)Pe^2 - St\,Pe - \frac{1}{2}St^2 = 0. \quad (23)$$

The valid solution to eqn (23) is

$$Pe = \frac{St(1+\sqrt{2/St-1})}{2(1/St-1)} \quad (24)$$

or, inversely,

$$St = -\frac{2}{3}Pe - \frac{2}{3}\frac{Pe^2}{g(Pe)} + \frac{g(Pe)}{3} \quad (25)$$

where,

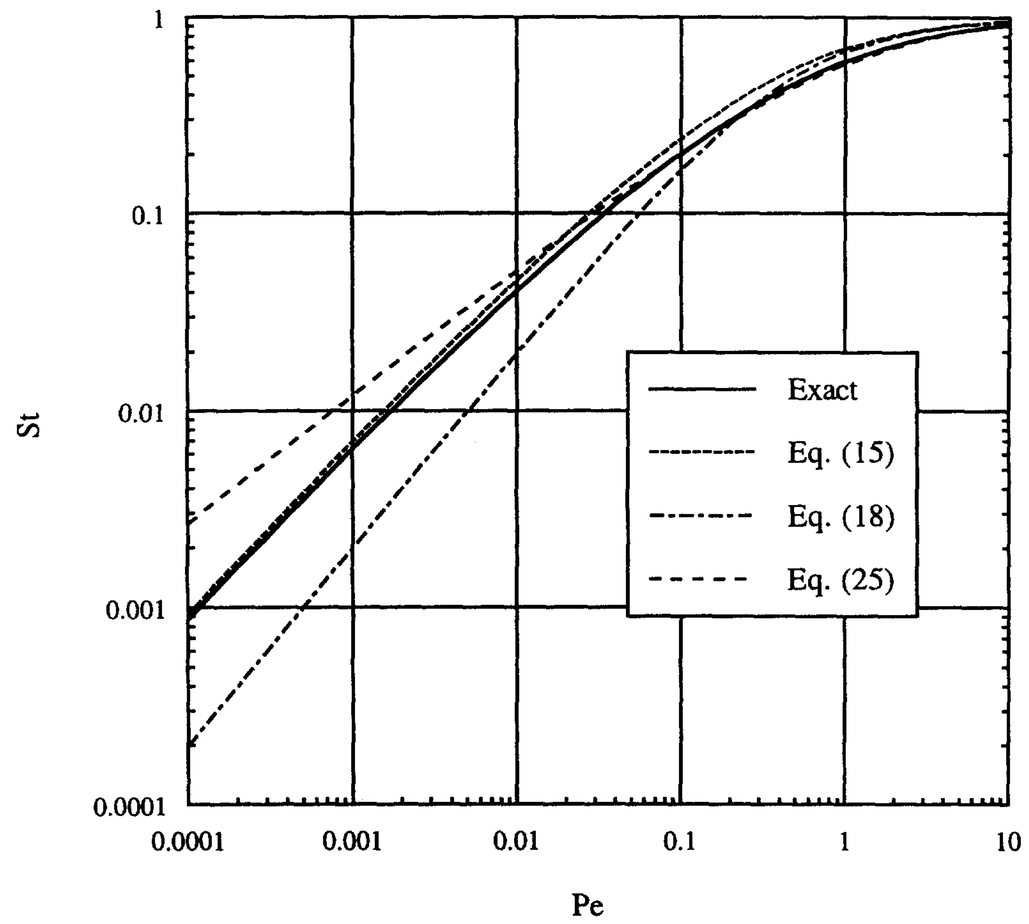

Fig. 2. Comparison of the present integral solutions with the exact Ivantsov solution.

$$g(Pe) = (27Pe^2 + 10Pe^3 + 3\sqrt{3}Pe^2\sqrt{27+20Pe+4Pe^2})^{1/3}. \quad (26)$$

When the Peclet number is larger than about 0.03, the above approximate solution matches the exact solution extremely well, as shown in Fig. 2. At smaller growth Peclet number, eqn (24) deviates considerably from the exact solution. The inverse form, eqn (24), is very useful in engineering calculations of solidification [8].

3. Conclusions

An integral analysis of the classical diffusion-controlled dendrite tip growth problem is carried out. Three temperature profiles are tried, resulting in three approximate solutions, eqns (15), (18) and (24). The approximate solution given by eqn (15) agrees best with the exact Ivantsov solution for the entire *Pe* number range. An even more accurate solution can be constructed by switching to eqn (24) for $Pe \geqslant 0.02$. The solution given by eqn (18) is in relatively poor agreement with the Ivantsov solution, but can be improved through curve fitting [8]. Equations (19) and (24) feature compact inverse forms, which are useful in engineering calculations.

The relative success of the present study provides some impetus to extend the integral method to the problem of dendrite tip growth with convection in the liquid [9].

Acknowledgement

This work was supported by the National Science Foundation (NSF) through grant no. CTS-9501389.

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